Mechanics of Solids (MEC2112/AEC2112) 2024-25

Tutorial 3 (Unit-2)

1. In a state of plane strain, the strain components associated with the $x - y$ axes are

$$\begin{aligned}\epsilon_{xx} &= 800 \times 10^{-6}\\ \epsilon_{yy} &= 100 \times 10^{-6}\\ \gamma_{xy} &= -800 \times 10^{-6}\end{aligned}$$

   Find the strain components in a new coordinate system $(x' - y')$ such that x' is at an angle of 45° with x.

2. The tensorial strain in a body is

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times 10^{-2}$$

   in a coordinate system $(x - y - z)$. Find the strain tensor in a new coordinate system $(x' - y' - z')$ such that x' is along $\hat{i} + \hat{j}$ and y' is in the direction of $-\hat{i} + \hat{j}$.

3. The state of stress in a body is

$$[\sigma] = \begin{bmatrix} 2 & 10 & 5 \\ 10 & 2 & 1 \\ 5 & 1 & 2 \end{bmatrix} \text{MPa.}$$

   Find the strain components if Young's modulus is 1 GPa and Poison's ratio is 0.25.

4. The displacement field for a body is described by

$$\begin{aligned} u(x, y, z) &= 0.05x + 0.01y \\ v(x, y, z) &= 0.01x + 0.02y. \\ w(x, y, z) &= 0.01z. \end{aligned}$$

   Find the stress tensor, if Young's modulus is 1 GPa and Poison's ratio is 0.25. [Hint: Use the relation between E, ν and λ, μ].

5. A cube of side 1 cm is confined by rigid walls on all sides and is heated by 30°C. What would be the stress (i.e., stress components) developed in the cube? Take α as 0.001 $/^o$C

6. If Young's modulus is 1 GPa and Poison's ratio is 0.25, obtain the stiffness matrix and compliance matrix for plane stress as well as plane strain cases. [Hint: The relation between E, ν and λ, μ may be useful].

① Let the transformed strain components are : $\epsilon_{x'x'}$, $\epsilon_{y'y'}$, $\gamma_{x'y'}$

$$\epsilon_{x'x'} = \frac{\epsilon_{xx}+\epsilon_{yy}}{2} + \frac{\epsilon_{xx}-\epsilon_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\epsilon_{y'y'} = \frac{\epsilon_{xx}+\epsilon_{yy}}{2} - \frac{\epsilon_{xx}-\epsilon_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\gamma_{x'y'} = -(\epsilon_{xx}-\epsilon_{yy})\sin 2\theta + \gamma_{xy}\cos 2\theta$$

$\epsilon_{xx} = 800\times10^{-6}$

$\epsilon_{yy} = 100\times10^{-6}$

$\gamma_{xy} = -800\times10^{-6}$

$\theta = 45°$

$$\epsilon_{x'x'} = \frac{800\times10^{-6}+100\times10^{-6}}{2} + \frac{800\times10^{-6}-100\times10^{-6}}{2}\times 0 + \frac{-800\times10^{-6}}{2}\times 1$$

$$\epsilon_{x'x'} = \frac{900\times10^{-6}}{2} + 0 + (-400\times10^{-6}) = 450\times10^{-6} - 400\times10^{-6}$$

$$= \boxed{50\times10^{-6}}$$

$$\epsilon_{y'y'} = \frac{800\times10^{-6}+100\times10^{-6}}{2} - \frac{800\times10^{-6}-100\times10^{-6}}{2}\times 0 - \frac{-800\times10^{-6}}{2}\times 1$$

$$\epsilon_{y'y'} = \frac{900\times10^{-6}}{2} - 0 + 400\times10^{-6} = \boxed{850\times10^{-6}}$$

$$\gamma_{x'y'} = -(800\times10^{-6} - 100\times10^{-6})\times 1 + (-800\times10^{-6})\times 0$$

$$\gamma_{x'y'} = (-700\times10^{-6}) + 0 = \boxed{-700\times10^{-6}}$$

② Strain Tensor

$$[\epsilon] = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times 10^{-2}$$

The new axes are :

x' is along $\hat{i}+\hat{j}$

y' is along $-\hat{i}+\hat{j}$

The dir of x' is $\frac{1}{\sqrt{2}}(1,1,0)$

The dir of y' is $\frac{1}{\sqrt{2}}(-1,1,0)$

z' axis will remain unchanged and orthogonal to both x' and y'

$\Rightarrow$ so it remains in dir $(0,0,1)$

<u>Rotation Matrix</u>

$$[R] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\epsilon'] = [R]^T[\epsilon][R]$$

$$[R]^T[\epsilon] = \begin{bmatrix} \frac{1}{\sqrt{2}}(2)+\frac{1}{\sqrt{2}}(-2) & \frac{1}{\sqrt{2}}(-2)+\frac{1}{\sqrt{2}}(1) & \frac{1}{\sqrt{2}}(1)+\frac{1}{\sqrt{2}}(1) \\ \frac{-1}{\sqrt{2}}(2)+\frac{1}{\sqrt{2}}(-2) & \frac{-1}{\sqrt{2}}(-2)+\frac{1}{\sqrt{2}}(1) & \frac{-1}{\sqrt{2}}(1)+\frac{1}{\sqrt{2}}(1) \\ 1 & 1 & 1 \end{bmatrix}$$

$$[R]^T[\epsilon] = \begin{bmatrix} 0 & -1/\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \times 10^{-2}$$

$$[\epsilon'] = \begin{bmatrix} 0\cdot\frac{1}{\sqrt2} + \left(\frac{-1}{\sqrt2}\right)\cdot\frac{1}{\sqrt2} + \sqrt2\cdot 0 & 0\cdot\frac{-1}{\sqrt2} + \left(\frac{-1}{\sqrt2}\right)\cdot\frac{1}{\sqrt2} + \sqrt2\cdot 0 & 0\cdot 0 + \left(\frac{-1}{\sqrt2}\right)\cdot 0 + \sqrt2\cdot 1 \\ -\sqrt2\cdot\frac{1}{\sqrt2} + \frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2} + 0\cdot 0 & -\sqrt2\cdot\frac{-1}{\sqrt2} + \frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2} + 0\cdot 0 & -\sqrt2\cdot 0 + \frac{1}{\sqrt2}\cdot 0 + 0\cdot 1 \\ 1\cdot\frac{1}{\sqrt2} + 1\cdot\frac{1}{\sqrt2} + 1\cdot 0 & 1\cdot\frac{-1}{\sqrt2} + 1\cdot\frac{1}{\sqrt2} + 1\cdot 0 & 1\cdot 0 + 1\cdot 0 + 1\cdot 1 \end{bmatrix}$$

$$[\epsilon'] = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \sqrt2 \\ -1 & \frac{1}{2} & 0 \\ \sqrt2 & 0 & 1 \end{bmatrix} \times 10^{-2}$$

$$[\epsilon'] = \begin{bmatrix} -0.5\times10^{-2} & -0.5\times10^{-2} & 1.41\times10^{-2} \\ -1.0\times10^{-2} & 0.5\times10^{-2} & 0 \\ 1.41\times10^{-2} & 0 & 1.0\times10^{-2} \end{bmatrix}$$

(3) $[\epsilon] = \frac{1}{E}\left([\sigma] - \nu\, tr(\sigma)[I]\right)$

Given :

$$[\sigma] = \begin{bmatrix} 2 & 10 & 5 \\ 10 & 2 & 1 \\ 5 & 1 & 2 \end{bmatrix} \text{MPa}$$

$E = 1\text{ GPa}$; $\nu = 0.25$

$tr(\sigma) = 2+2+2 = 6\text{ MPa}$

$$\boxed{[\epsilon] = \begin{bmatrix} 0.0005 & 0.01 & 0.005 \\ 0.01 & 0.0005 & 0.001 \\ 0.005 & 0.001 & 0.0005 \end{bmatrix}}$$

$[\epsilon] = \frac{1}{E}\left([\sigma] - \nu\, tr(\sigma)[I]\right)$

$$[\epsilon] = \frac{1}{1000}\left(\begin{bmatrix} 2 & 10 & 5 \\ 10 & 2 & 1 \\ 5 & 1 & 2 \end{bmatrix} - 0.25\times 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$[\epsilon] = \frac{1}{1000}\left(\begin{bmatrix} 2 & 10 & 5 \\ 10 & 2 & 1 \\ 5 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\right)$$

$$= \frac{1}{1000}\begin{bmatrix} 0.5 & 10 & 5 \\ 10 & 0.5 & 1 \\ 5 & 1 & 0.5 \end{bmatrix}$$

(4) $u(x,y,z) = 0.05x + 0.01y$

$v(x,y,z) = 0.01x + 0.02y$

$w(x,y,z) = 0.01z$

$\epsilon_{xx} = \frac{\partial u}{\partial x}$

$\epsilon_{yy} = \frac{\partial v}{\partial y}$

$\epsilon_{zz} = \frac{\partial w}{\partial z}$

$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$

$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$

$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0.01+0.01)$

$= 0.01$

$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}(0+0) = 0$

$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(0+0) = 0$

$$[\epsilon] = \begin{bmatrix} 0.05 & 0.01 & 0 \\ 0.01 & 0.02 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$[\sigma] = \lambda\, tr(\epsilon)[I] + 2\mu[\epsilon]$

$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$

$\boxed{\mu = \frac{E}{2(1+\nu)}} = \frac{1000}{2(1+0.25)} = \frac{1000}{2.5}$

$= 400\text{ MPa}$

$$\lambda = \frac{1000 \times 0.25}{(1+0.25)(1-2\times 0.25)} = \frac{250}{1.25 \times 0.5} = \frac{250}{0.625} = 400 \text{ MPa}$$

$$tr(\epsilon) = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0.05 + 0.02 + 0.01 = 0.08$$

$$[\sigma] = 400 \times 0.08 [I] + 2 \times 400 \times \begin{bmatrix} 0.05 & 0.01 & 0 \\ 0.01 & 0.02 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} + \begin{bmatrix} 40 & 8 & 0 \\ 8 & 16 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\boxed{[\sigma] = \begin{bmatrix} 72 & 8 & 0 \\ 8 & 48 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa}}$$

(5)

$$\alpha = 0.001 \, ^\circ C^{-1}$$

$$\Delta T = 30^\circ C$$

length of cube $\Rightarrow L = 1cm$

$$\epsilon_{Thermal} = \alpha \Delta T$$

$$= 0.001 \times 30$$

$$= 0.03$$

$$\sigma = E \, \epsilon_{Thermal}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{thermal}$$

$$\sigma_{thermal} = E \, \epsilon_{thermal}$$

$$\Rightarrow [\sigma] = \begin{bmatrix} \sigma_{thermal} & 0 & 0 \\ 0 & \sigma_{thermal} & 0 \\ 0 & 0 & \sigma_{thermal} \end{bmatrix}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = E \times 0.03$$

$$\sigma_{thermal} = 1000 \times 0.03$$

$$= 30 \text{ MPa}$$

$$[\sigma] = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ MPa}$$

$$\boxed{\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 30 \text{ MPa}}$$

(6)

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)} = \frac{1000}{2(1+0.25)} = 400 \text{ MPa}$$

$$\lambda = \frac{1000 \times 0.25}{(1+0.25)(1-2\times 0.25)} = \frac{250}{1.25 \times 0.5} = \frac{250}{0.625}$$

$$= 400 \text{ MPa}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{\mu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$[C] = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{\mu}{2} \end{bmatrix}$$

$E = 1000$ MPa and $\nu = 0.25$

$$\frac{E}{1-\nu^2} = \frac{1000}{1-(0.25)^2} = \frac{1000}{0.9375} = 1066.67 \text{ MPa}$$

$$\frac{\nu E}{1-\nu^2} = \frac{0.25 \times 1000}{0.9375} = \frac{250}{0.9375} = 266.67 \text{ MPa}$$

$$\frac{\mu}{2} = \frac{400}{2} = 200 \text{ MPa}$$

$$[C] = \begin{bmatrix} 1066.67 & 266.67 & 0 \\ 266.67 & 1066.67 & 0 \\ 0 & 0 & 200 \end{bmatrix} \text{MPa}$$

Plain Strain Case:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{E}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{\mu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$[C] = \begin{bmatrix} \frac{E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{E}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{\mu}{2} \end{bmatrix}$$

$E = 1000\,\text{MPa}$ and $\nu = 0.25$,

$$\frac{E}{(1+\nu)(1-2\nu)} = \frac{1000}{(1+0.25)(1-2\times 0.25)} = \frac{1000}{1.25\times 0.5} = \frac{1000}{0.625}$$

$$= 1600\,\text{MPa}$$

$$\frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{0.25\times 1000}{0.625} = \frac{250}{0.625} = 400\,\text{MPa}$$

$$\frac{\mu}{2} = 200\,\text{MPa}$$

$$[C] = \begin{bmatrix} 1600 & 400 & 0 \\ 400 & 1600 & 0 \\ 0 & 0 & 200 \end{bmatrix}$$

$$[S] = [C]^{-1}$$

$$[S] = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{\mu} \end{bmatrix}$$

$$\frac{1-\nu^2}{E} = \frac{1-(0.25)^2}{1000} = \frac{0.9375}{1000} = 0.0009375$$

$$-\frac{\nu}{E} = -\frac{0.25}{1000} = -0.00025$$

$$\frac{1}{\mu} = \frac{1}{400} = 0.0025$$

$$[S] = \begin{bmatrix} 0.0009375 & -0.00025 & 0 \\ -0.00025 & 0.0009375 & 0 \\ 0 & 0 & 0.0025 \end{bmatrix} \text{MPa}^{-1}$$

$$[S] = [C]^{-1}$$

$$[S] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{\mu} \end{bmatrix}$$

$$\frac{1}{E} = \frac{1}{1000} = 0.001$$

$$-\frac{\nu}{E} = \frac{-0.25}{1000} = -0.00025$$

$$\frac{1}{\mu} = \frac{1}{400} = 0.0025$$

$$[S] = \begin{bmatrix} 0.001 & -0.00025 & 0 \\ -0.00025 & 0.001 & 0 \\ 0 & 0 & 0.0025 \end{bmatrix} MPa^{-1}$$

Stiffness matrix for plane stress

$$[C] = \begin{bmatrix} 1066.67 & 266.67 & 0 \\ 266.67 & 1066.67 & 0 \\ 0 & 0 & 200 \end{bmatrix} MPa$$

Stiffness matrix for plane strain

$$[C] = \begin{bmatrix} 1600 & 400 & 0 \\ 400 & 1600 & 0 \\ 0 & 0 & 200 \end{bmatrix} MPa$$

Compliance matrix for plane stress:

$$[S] = \begin{bmatrix} 0.0009375 & -0.00025 & 0 \\ -0.00025 & 0.0009375 & 0 \\ 0 & 0 & 0.0025 \end{bmatrix} MPa^{-1}$$

Compliance matrix for plane strain:

$$[S] = \begin{bmatrix} 0.001 & -0.00025 & 0 \\ -0.00025 & 0.001 & 0 \\ 0 & 0 & 0.0025 \end{bmatrix} MPa^{-1}$$